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Development of Stresses in Cohesionless Poured Sand

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The pressure distribution beneath a conical sandpile, created by pouring sand from a point source onto a rough rigid support, shows a pronounced minimum below the apex ('the dip'). Recent work of the authors has attempted to explain this phenomenon by invoking local rules for stress propagation that depend on the local geometry, and hence on the construction history, of the medium. We discuss the fundamental difference between such approaches, which lead to hyperbolic differential equations, and elastoplastic models, for which the equations are elliptic within any elastic zones present. In the hyperbolic case, the stress distribution at the base of a wedge or cone (of given construction history), on a rough rigid support, is uniquely determined by the body forces and the boundary condition at the free (upper) surface. In simple elastoplastic treatments one must in addition specify, at the base of the pile, a displacement field (or some equivalent data). This displacement field appears to be either ill-defined, or defined relative to a reference state whose physical existence is in doubt. Insofar as their predictions depend on physical factors unknown and outside experimental control, such elastoplastic models predict that the observations should be intrinsically irreproducible. This view is not easily reconciled with the existing experimental data on conical sandpiles, which we briefly review. Our hyperbolic models are based instead on a physical picture of the material, in which (a) the load is supported by a skeletal network of force chains ("stress paths") whose geometry depends on construction history; (b) this network is 'fragile' or marginally stable, in a sense that we define. Although perhaps oversimplified, these assumptions may lie closer to the true physics of poured cohesionless grains than do those of conventional elastoplasticity. We point out that our hyperbolic models can nonetheless be reconciled with elastoplastic ideas by taking the limit of an extremely anisotropic yield condition.

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1. Introduction

Recently, a new strategy for the modelling of stress propagation in static cohesionless granular media was developed (Bouchaud *et al.* 1995; Wittmer *et al.* 1996, 1997*a*, *b*). The medium is viewed as an assembly of rigid particles held up by friction. The static indeterminacy of frictional forces within the assembly is circumvented by the assumption of certain local *constitutive relations* (c.r.'s) among components of the stress tensor.† These are assumed to encode the network of contacts in the granular packing geometry; they therefore depend explicitly on the way in which the medium was made – its *construction history.* The task is then to postulate and/or justify physically suitable c.r.'s among stresses, of which only one (the *primary* c.r.) is required for systems with two dimensional symmetry, such as a wedge of sand; for a three dimensional symmetric assembly (the conical sandpile) a secondary c.r. is also needed.

Among the primary constitutive relations of Wittmer *et al.* (1996, 1997*a*) are a certain class (called the 'oriented stress linearity' or OSL models) which have simplifying features. Indeed, in two-dimensional geometries these combine with the stress continuity equation to give a wave equation for stress propagation, in which the horizontal and vertical directions play the role of spatial and temporal coordinates respectively (Bouchaud *et al.* 1995). A distinguishing feature of the OSL models is that the *characteristic rays for stress propagation* (analagous to light or sound rays in ordinary wave propagation) are then fixed by the construction history: they do not change direction under subsequent reversible loading. (Irreversible loadings, which can in these models be infinitesimal, are discussed in Section 6 below.) As discussed by Bouchaud *et al.* (1998), the characteristics of the differential equation can be viewed as representing, in the continuum, the mean behaviour of 'force chains' or 'stress paths' in the material (Dantu 1967; Liu *et al.* 1995; Thornton & Sun 1994).

Of the OSL models, a particularly appealing member, with special symmetry properties, is called the 'fixed principal axes' (FPA) model. This has the additional property that the characteristics everywhere coincide in orientation with the principal axes of the stress tensor. The FPA model therefore supposes that these principal axes have an orientation fixed at the time of burial. ‡ This is arguably the simplest possible choice for a history-dependent c.r. among stresses. For the case of a sandpile in which grains are deposited by surface avalanches, which we presume to apply for a conical pile constructed from a point source (though see Section 5c below), the orientation of the major axis at burial is constant, and known from the fact that the free surface in such a pile must be a yield surface. The resulting constitutive equation among stresses, for the sandpile ge-

† In solid mechanics the term 'constitutive relation' normally refers to a material-dependent equation relating stress and strain. In fluid mechanics one has instead equations relating stress and (in the general case of a viscoelastic fluid) strain-rate history. Our models of granular media entail equations relating stress components to one another, in a manner that we take to depend on the construction history of the material. Clearly such equations describe constitutive properties of the medium: they relate its state of stress to other discernable features of its physical state. We see no alternative to the term 'constitutive relations' for such equations. The same equations could, of course, be obeyed by some solutions of models whose constitutive definition was quite different; in that context they would not be c.r.'s.

‡ Gudehus (1974) has previously used a related idea, that the principal axes should be locally specified as inputs to the stress continuity equations. This he employs as a calculation method for generating a variety of stress distributions based on 'gutem statischen Gefühl'.

ometry, then has a singularity at the centre of a cone or wedge; this is physically admissible since the centreline separates material which has avalanched to the left from material which has avalanched to the right. This singularity leads to an 'arching' effect, as previously invoked to explain the stress-dip by Edwards & Oakeshott (1987) and others (Trollope 1968; Trollope & Burman 1980).

The OSL models were developed to explain experimental data on the stress distribution beneath a conical sandpile, built by surface avalanches of sand, poured from a point source onto a rough, rigid support (Smid & Novosad 1981; Jokati & Moriyama 1979; Brockbank *et al.* 1997). Such data shows unambiguously the presence of a minimum ('the pressure dip') in the vertical normal stress below the apex of the pile. With a plausible choice of secondary c.r. (of which several were tried, with only minor differences resulting), the FPA case, in particular, was found to give a fairly good quantitative account of the data of Smid & Novosad (1981), and of Brockbank *et al.* (1997); see Fig. 1. This is remarkable, in view of the radical simplicity of the assumptions made.

We accept, of course, that such models may be valid only a limited regime in some larger parameter space. For example, since strain variables are not introduced, these models cannot of themselves examine the crossover to conventional elastic or elastoplastic behaviour that must presumably arise when the applied stresses are significant on the scale of the elastic modulus of the grains themselves. Some further remarks on this crossover, in the context of anisotropic elastoplasticity, are made in Section 3d .

In this paper we discuss the physical content of our general modelling approach (of which the FPA model is one example), based on local stress propagation rules that depend on construction history, as encoded in constitutive relations among stresses. In particular we contrast the approach with more conventional ideas – especially the ideas of elastoplasticity. For simplicity, our mathematical discussion is mainly limited to two dimensions (although our models were developed to describe three-dimensional piles) and to the simplest, isotropic forms of elastoplastic theory. The discussion aims to sharpen some conceptual issues. These concern not the details of particular models, but the general question of what *sort* of description we should aspire to: what sort of information do we need as modelling input, and what can be predicted as output? An equally important (and closely related) question is, what are the control variables in an experiment that must be specified to ensure reproducible behaviour, and what are the observables that can then be measured to depend on these? For experiments on sandpiles (briefly reviewed in Section 5) we believe these to be open physics questions, and to challenge some widely held assumptions of the applicability of traditional elastoplastic modelling strategies to cohesionless poured grains.

The proposal that granular assemblies under gravity cannot properly be described by the ideas of conventional elastoplasticity has been opprobiously dismissed in some quarters: we stand accused of ignoring all that is 'long and widely known' among geotechnical engineers (Savage 1997). However, we are not the first to put forward such a subversive proposal. Indeed workers such as Trollope (1968) and Harr (1977) have long ago developed ideas of force transfer rules among discrete particles, not unrelated to our own approaches, which yield continuum equations quite unlike those of elastoplasticity. More recently, dynamical *hypoplastic* continuum models have been developed (Kolymbas 1991, Kolymbas and Wu 1993) which, as explained by Gudehus (1997) describe an 'anelastic be-

haviour without [the] elastic range, flow conditions and flow rules of elastoplasticity'. Our own models, though not explicitly dynamic, are similarly anelastic, as we discuss in Section 6. They should perhaps be classified as hypoplastic models, although their relation to extremely *anisotropic* elastoplastic models is examined in Section 3d below.

2. Continuum models of cohesionless granular matter

We start by reviewing (in their simplest forms) some well-known modelling approaches based on rigid-plastic and elastoplastic ideas. This is followed by a brief summary of the mathematical content of the FPA model and its relatives.

(*a*) *Stress continuity and the Coulomb inequality*

The equations of stress continuity express the fact that, in static equilibrium, the forces acting on a small element of material must balance. For a conical pile of sand we have, in $d = 3$ dimensions,

$$\partial_r \sigma_{rr} + \partial_z \sigma_{zr} = \beta(\sigma_{\chi\chi} - \sigma_{rr})/r \tag{2.1}$$

$$\partial_r \sigma_{rz} + \partial_z \sigma_{zz} = g - \beta\sigma_{rz}/r \tag{2.2}$$

$$\partial_\chi \sigma_{ij} = 0 \tag{2.3}$$

where $\beta = 1$. Here z, r and χ are cylindrical polar coordinates, with z the downward vertical. We take $r = 0$ as a symmetry axis, so that $\sigma_{r\chi} = \sigma_{z\chi} = 0$; g is the force of gravity per unit volume; σ_{ij} is the usual stress tensor which is symmetric in i, j. The equations for $d = 2$ are obtained by setting $\beta = 0$ in (2.1,2.2) and suppressing (2.3). These describe a wedge of constant cross section and infinite extent in the third dimension. They also describe a layer of grains in a thin, upright Hele-Shaw cell, but only if the wall friction is negligible.

The Coulomb inequality states that, at any point in a cohesionless granular medium, the shear force acting across any plane must be smaller in magnitude than $\tan\phi$ times the compressive normal force. Here ϕ is the angle of friction, a material parameter which, in simple models, is equal to the angle of repose. We accept this here, while noting that (i) ϕ in principle depends on the texture (or fabric) of the medium and hence on its construction history; (ii) for a highly anisotropic packing, the existence of an orientation-independent ϕ is questionable (see Section 3d); (iii) the identification of ϕ with the repose angle ignores some complications such as the Bagnold hysteresis effect.

(*b*) *Rigid-plastic models*

The model that Wittmer *et al.* (1996, 1997*a*) refer to as "incipient failure everywhere" (IFE), is more commonly called the rigid-plastic model. It postulates that the Coulomb condition is everywhere obeyed *with equality* (Nedderman 1992). That is, through every point in the material there passes some plane across which the shear force is exactly $\tan\phi$ times the normal force. By assuming this, the IFE model allows closure (modulo a sign ambiguity discussed below) of the equations for the stress without invocation of an elastic strain field. The IFE model has therefore as its 'constitutive relation' (Wittmer *et al.* 1997*a*):

$$\sigma_{rr} = \sigma_{zz}\,\frac{1}{\cos^2\phi}\left[\sin^2\phi + 1 \pm 2\sin\phi\,\sqrt{1 - (\cot\phi\;\;\sigma_{zr}/\sigma_{zz})^2}\right] \tag{2.4}$$

whereas the Coulomb inequality requires only that σ_{rr} lies between the two values ($\pm$) on the right.

The postulate that a Coulombic slip plane passes through each and every material point is not usually viewed as being accurate in itself; the rigid-plastic model is more often proposed as a way of generating certain 'limit-state' solutions to an underlying elastoplastic model. In the simplest geometries these solutions correspond to taking the $-$ or $+$ sign in (2.4). It is a simple exercise to show that for a sandpile at its repose angle, only one solution of the resulting equations exists in which the sign choice is everywhere the same. This requires the negative root (conventionally referred to as an 'active' solution) and it shows a hump, not a dip, in the vertical normal stress beneath the apex. Savage (1997), however, draws attention to a 'passive' solution, having a pronounced dip beneath the apex. This solution actually contains a pair of matching planes between an inner region where the positive root of (2.4) is taken, and an outer region where the negative is chosen.

In principle there is more than one such 'passive' solution. For example one can seek an IFE solution in which all stress components are continuous across the matching plane. This requires a discontinuity in the gradients of the stresses at the centreline (see Fig. 2). The latter does not contradict Eq. (2.4), although it might be thought undesirable on other grounds (for example if the IFE equation is thought to bound the behaviour of a simple elastoplastic body, for which the resulting displacement fields might not be admissible). An alternative, which avoids this, is to instead have a discontinuity of the normal stress parallel to the matching plane itself. This gives a second passive solution (Savage 1997,1998). These solutions do not exhaust the repertoire of IFE solutions for the sandpile: there is no physical principle that limits the number of matching surfaces. By adding extra ones, a very wide variety of results can be achieved.

The emphasis placed on the rigid-plastic approach, at least in some parts of the literature, seems to rest on a misplaced belief that the limit-state solutions can be 'generally regarded as bounds between which other states can exist, *i.e.*, when the material is behaving in an elastic or elastoplastic manner' (Savage 1997). A simple counterexample is shown in Fig. 2. This shows the active and two passive solutions (as defined above) for a two-dimensional pile (wedge), along with an elementary elastoplastic solution as presented recently by Cantelaube & Goddard (1997) and earlier by Samsioe (1955). The latter is piecewise linear with no singularity in the displacement field on the central axis; it happens to coincide mathematically with the solution of a simple hyperbolic model (Bouchaud *et al.* 1995) for the same geometry (there is no stress dip in this particular model). Clearly the vertical normal stress does not lie everywhere between that of the active and passive IFE solutions, which are therefore not bounds.

(*c*) *Elastoplastic models*

In two dimensions at least, it has been argued that the pressure dip can be explained within a simple conventional elastoplastic modelling approach. This is certainly possible if the base is allowed to sag slightly (Savage 1997). Here, however, we are concerned with piles built on a rough, rigid support. Even in this case, it has been argued that results similar to those of the FPA model can be obtained (Cantelaube & Goddard 1997). The simplest elastoplastic models assume a material in which a perfectly elastic behaviour at small strains is conjoined onto

perfect plasticity (the Coulomb condition with equality) at larger ones. In such an approach to the sandpile, an inner elastic region is matched, effectively by hand, onto an outer plastic one. In the inner elastic region the stresses obey the Navier equations, which follow from those of Hookean elasticity by elimination of the strain variables. The corresponding strain field is not discussed, but tacitly treated as infinitesimal, since the high modulus limit is taken. Howevever, for FPA-like solutions, which show a cusp in the vertical stress on the centreline, the displacement shows singular features which are not easily reconciled with a purely elastoplastic interpretation. The fact that the plastic zone is introduced ad-hoc also has drawbacks – for example it is hard to explain the continued presence of such a zone if the angle of the pile is reduced to slightly below the friction angle ϕ (to allow for the Bagnold hysteresis effect, say). In OSL approaches, an outer zone is not assumed but predicted, and remains present in this case, although the material in this zone is no longer at incipient failure.

The existence of FPA-like solutions to simple elastoplastic models in three dimensions, on a non-sagging support, remains very much in doubt. But in any case, our objections to the elastoplastic approach to modelling sandpiles lie at a more fundamental level. Specifically it appears that, to make unambiguous predictions for the stresses in a sandpile, these models require boundary information that has no obvious physical meaning or interpretation. We return to this physics problem in Section 3b .

(*d*) *Local-rule models of stress propagation*

In the FPA model (Wittmer *et al.* 1996, 1997*a*) one hypothesizes that, in each material element, the orientation of the stress ellipsoid became 'locked' into the material texture at the time when it last came to rest, and does not change in subsequent loadings (unless forced to: see Section 6). This is a bold, simplifying assumption, and it may indeed be far too simple, but it exemplifies the idea of having a local rule for stress propagation that depends explicitly on construction history. For the sandpile geometry, where the material comes to rest on a surface slip plane, this constitutive hypothesis leads to the following relation among stresses:

$$\sigma_{rr} = \sigma_{zz} - 2\tan\phi\ \sigma_{zr} \tag{2.5}$$

where ϕ is the angle of repose. Eq.(2.5) is algebraically specific to the case of a pile created from a point source by a series of avalanches along the free surface.

A consequence of Eq. (2.5), for a pile at repose, is that the major principal axis everywhere bisects the angle between the vertical and the free surface. It should be noted that in cartesian coordinates, the FPA model reads :

$$\sigma_{xx} = \sigma_{zz} - 2\,\mathrm{sign}(x)\tan\phi\ \sigma_{xz} \tag{2.6}$$

where x is horizontal. From Eq. (2.6), the FPA constitutive relation is seen to be discontinuous on the central axis of the pile: the local texture of the packing has a singularity on the central axis which is reflected in the stress propagation rules of the model. The paradoxical requirement, on the centreline, that the principal axes are fixed simultaneously in two different directions has a simple resolution: the stress tensor turns out to be isotropic there.

The FPA model is one of the larger class of OSL models in which the primary

constitutive relation (in the sandpile geometry) is, in Cartesians

$$\sigma_{xx} = \eta\sigma_{zz} + \mu\,\mathrm{sign}(x)\ \sigma_{xz} \tag{2.7}$$

with η, μ constants. As explained Wittmer *et al.* (1997*a*), these models (in two dimensions) yield hyperbolic equations that have *fixed characteristic rays* for stress propagation. (Unless $\mu = 0$, these are asymmetrically disposed about the vertical axis, and invert discontinuously at the centreline $x = 0$.) The constitutive property that OSL models describe is that these characteristic rays, or force chains, have orientations that are 'locked in' on burial of an element. The boundary condition, that the free surface of a pile at its angle of repose ϕ is a slip plane, yields one equation linking η and μ to ϕ; thus the OSL scheme represents a one-parameter family of models. Note that, as soon as η is not exactly unity (the FPA case) the orientation of the principal axes rotates smoothly as one passes through the centreline of the pile. The assumption of fixed principal axes, though appealing, is thus rather delicate, and arguably much less important than the idea of fixed characteristics, since these represent the average geometry of force chains in the medium. The experimental data (Fig. 1) supports models in the OSL family with η close, but perhaps not exactly equal, to unity.

Note that, unless the OSL parameter is chosen so that $\mu = 0$, a constitutive singularity on the central axis remains. The case $\mu = 0$ corresponds to one studied earlier by Bouchaud *et al.* (1995); this 'BCC' model is the only member of the OSL family to have characteristics symmetric about the vertical. (Their angles $\pm\theta$ to the vertical obey $\tan^2\theta = c_0^2 = \eta$.) This latter model could be called a 'local Janssen model' in that it assumes *local* proportionality of horizontal and vertical compressive stresses – an assumption which, when applied *globally* to average stresses in a silo, was first made by Janssen (1895).

The local-rule models just discussed do not account for the presence of 'noise' or randomness in the local texture. Such effects have been studied by Claudin *et al.* (1998), and, if the noise level is not too large, lead at large length scales to effective wavelike equations with additional gradient terms giving a diffusive spreading of the characteristic rays. The limit where the diffusive term dominates corresponds to a *parabolic* differential equation (Harr 1977), similar to those arising in scalar force models (Liu *et al.* 1995) which have, in effect, a single downward characteristic (so the main interest lies in the diffusive spreading). It is possible that under extreme noise levels this picture changes again, although this conclusion is based on assumptions about the noise itself that may not be valid (Claudin *et al.* 1998). The discussions that follow therefore apply to local-rule models with moderate, but perhaps not extreme, noise.

Note finally that the fact that two continuum models, based on different constitutive hypotheses, can give identical results for the stresses in some specified geometry, obviously does not mean that the models are equivalent. (Equivalence requires, at least, that the Green function of the two models is also the same.) Thus, for example, models such as FPA are not equivalent to Trollope's model of "clastic discontinua" (Trollope 1968, Trollope and Burman 1980). In Appendix B we outline the relationship between our work and the marginal packing models studied by Ball & Edwards (to be published), Huntley (1993), Hong (1993), Bagster (1978) and Liffmann *et al.* (1992), as well as the work of Trollope (1956,1968).

3. Strain and displacement variables

In the FPA model and its relatives, strain variables are not considered. A partial justification for this was given by Wittmer *et al.* (1997*a*), namely that the experimental data obey a form of radial stress-field (RSF) scaling: the stress patterns observed at the base are the same shape regardless of the overall height of the pile. Formally one has for the stresses at the base

$$\sigma_{ij} = gh\, s_{ij}(r/ch) \tag{3.1}$$

where h is the pile height, $c = \cot\alpha$ and s_{ij} a reduced stress: α is the angle between the free surface and the horizontal so that for a pile at repose, $\alpha = \phi$. This form of RSF scaling, which involves only the forces at the base (Evesque 1997), might be called the 'weak form' and is distinct from the 'strong form' in which Eq. (3.1) is obeyed also with z (an arbitrary height from the apex) replacing h (the overall height of the pile).

This scaling implies that there is no characteristic length-scale. Since elastic deformation introduces such a length-scale (the length-scale over which an elastic pile would sag under gravity) the observation of RSF scaling to experimental accuracy suggests that elastic effects *need not be considered explicitly.* We accept however (correcting Wittmer *et al.* 1997*a*) that this does not of itself rule out elastic or elastoplastic behaviour which, at least in the limit of large modulus, can also yield equations for the stress from which the bulk strain fields cancel. Note that it is tempting, but entirely wrong, to assume that a similar cancellation occurs at the boundaries of the material; we return to this below (Section 4).

The cancellation of bulk strain fields in elastoplastic models is convenient since there appears to be no clear definition of strain or displacement for piles constructed by pouring sand grainwise from a point source. To define a physical displacement or strain field, one requires a reference state. In (say) a triaxial strain test (see e.g. Wood 1990) an initial state is made by some reproducible procedure, and strains measured from there. The elastic part is identifiable in principle, by removing the applied stresses (maintaining an isotropic pressure) and seeing how much the sample recovers its shape. In contrast, a pile constructed by pouring grains onto its apex cannot convincingly be described in terms of the plastic and/or elastic deformation from some initial reference state of the same continuous medium: the corresponding experiments are unrealizable. Even were the load (which consists purely of gravity) to be removed, the resulting unstrained body would comprise grains floating freely in space with no definite positions. It is unsatisfactory to define a strain or displacement field with respect to such a body. The problem occurs whenever the solidity of the body itself *only* arises because of the load applied. A similar situation occurs, for example, in colloidal suspensions that flow freely at small shear stresses but (by jamming) can support larger ones indefinitely (Cates *et al.*, to be published).

Although one cannot uniquely define the strain in a granular assembly under gravity, it may of course be possible to define *incremental* strains in terms of the displacement of grains when a small load is added. However, the range of stress increments involved might in practice be negligible (Kolymbas 1991; Gudehus 1997).

4. The role of boundary conditions

(*a*) *Boundary conditions in hyperbolic (and parabolic) models*

Models that assume local constitutive equations among stresses (including all OSL models, and also the IFE or rigid-plastic model) provide hyperbolic differential equations for the stress field. Accordingly, if one specifies a zero-force boundary condition at the free (upper) surface of a wedge, then any perturbation arising from a small extra body force (a 'source term' in the equations) propagates along two characteristics passing through this point. (In the OSL models these characteristics are, moreover, straight lines.) Therefore the force at the base can be found simply by summing contributions from all the body forces as propagated along two characteristic rays onto the support; the sandpile problem is, within the modelling approach by Bouchaud *et al.* (1995) and Wittmer *et al.* (1996, 1997*a*), mathematically well-posed.

Note that in principle, one could have propagation also along the 'backward' characteristics (see Fig. 3(a)). This is forbidden since these cut the free surface; any such propagation can only arise in the presence of a nonzero surface force, in violation of the boundary conditions. Therefore the fact that the propagation occurs only along downward characteristics is not related to the fact that gravity acts downward; it arises because we know already the forces acting at the free surface (they are zero). Suppose we had instead an inverse problem: a pile on a bed with some unspecified overload at the top surface, for which the forces acting at the base had been measured. In this case, the information from the known forces could be propagated along the *upward* characteristics to find the unknown overload. More generally, in OSL models of the sandpile, each characteristic ray will cut the surface of a (convex) patch of material at two points. Within these models, the sum of the forces tangential to the ray at the two ends must be balanced by the tangential component of the body force integrated along the ray (see Fig. 3(b)). We discuss this physics (that of force chains) in Section 6a.

In three dimensions, the mathematical structure of these models is somewhat altered (Bouchaud *et al.* 1995), but the conclusions are basically unaffected. Note however that for different geometries, such as sand in a bin, the problem is not well-posed even with hyperbolic equations, unless something is known about the interaction between the medium and the sidewalls. Ideally one would like an approach in which sidewalls and base were treated on an equal basis; this is the subject of ongoing research. Note also that the essential character of the boundary value problem is not altered when appropriate forms of randomness are introduced. For although the response to a point force is now spread about the two characteristics, even in the parabolic limit (where the underlying straight rays are effectively coincident and only spreading remains) the sandpile boundary value problem remains well posed.

(*b*) *The physics of elastic indeterminacy*

The well-posedness of the sandpile does not extend to models involving the elliptic equations for an elastic body. For such a material, the stresses throughout the body can be solved only if, at all points on the boundary, either the force distribution or a displacement field is specified (Landau & Lifshitz 1986). Accordingly, once the zero-stress boundary condition is applied at the free surface, nothing can in principle be calculated unless either the forces or the displacements

at the base are already known (and the former amounts to specifying in advance the solution of the problem). From an elastoplastic perspective, it is clearly absurd to try to calculate the forces on the support, which are the experimental data, without some further information about what is happening at the bottom boundary. We have called this the problem of 'elastic indeterminacy' (Bouchaud *et al.* 1998) although perhaps 'elastic ill-posedness' would be a better term. The problem does not arise from any uncertainty about what to do mathematically: one should specify a displacement field at the base. Difficulties nonetheless arise if, as we argued above, no physical significance can be attributed to this displacement field for cohesionless poured sand.

To give a specific example, consider the static equilibrium of an elastic cone or wedge of finite modulus resting on a completely rough, rigid surface (which one could visualize as a set of pins; Fig. 4). Starting from any initial configuration, another can be generated by pulling and pushing parts of the body horizontally across the base (*i.e.*, changing the displacements there); if this is rough, the new state will still be pinned and will achieve a new static equilibrium. This will generate a stress distribution, across the supporting surface and within the pile, that differs from the original one. If a large enough modulus is now taken (at fixed forces), this procedure allows one to generate arbitrary differences in the stress distribution while generating neither appreciable distortions in the shape of the cone, nor any forces at its free surface. Analogous remarks apply to any simple elastoplastic theory of sandpiles, in which an elastic zone, in contact with part of the base, is attached at matching surfaces to a plastic zone.

In contrast, experimental reports (reviewed in Section 5) indicate that for sandpiles on a rough rigid support, the forces on the base can be measured reproducibly. They also suggest that these forces, although subject to statistical fluctuations on the scale of several grains, do not vary too much from one pile to another, at least among piles constructed in the same way (e.g., by avalanches from a point source), from the same material. This argues strongly against the idea that such forces in fact depend on a basal displacement field, which is determined either by the whim of the experimentalist, or by some as-yet unspecified physical mechanism acting at the base of the pile. Note that basal sag is *not* a candidate for the missing mechanism, since it does not resolve the elastic indeterminacy in these models; the latter arises primarily from the *roughness*, rather than the rigidity, of the support. Note also, however, that elastic indeterminacy can be alleviated in practice if the elastoplastic model is sufficiently anisotropic; we return to this point in Section 3d.

Evesque (private communication), unlike many authors, does confront the issue of elastic indeterminacy and seemingly concludes that the experimental results *are and must be indeterminate*; he argues that the external forces acting on the base of a pile can indeed be varied at will by the experimentalist, without causing irreversible rearrangements of the grains (see also Evesque & Boufellouh 1997). To what extent this viewpoint is based on experiment, and to what extent on an implicit presumption in favour of elastoplastic theory, is to us unclear.

(*c*) *Displacements to the rescue?*

Let us boldly suppose, then, that the experimental data is meaningful and reproducible, at least as far as the global, 'coarse-grained' features of the observations are concerned. (Noise effects at the level of individual grains may in

contrast be exquisitely sensitive to temperature and other poorly-controlled parameters; Claudin & Bouchaud 1997.) Adherents to traditional elastoplastic models then have three choices. The first is to consider the possibility that, after all, the problem of cohesionless poured sand may be better described by quite different governing equations from those of simple elastoplasticity. This possibility, which represents our own view, has certainly been suggested before. For example, hypoplastic models in which there is negligible elastic range (Gudehus 1997; Kolymbas 1991, Kolymbas and Wu 1993) do not suffer from elastic indeterminacy.

The second choice is to postulate various additional constraints, so as to eliminate some of the infinite variety of solutions that elastoplastic models allow (unless basal displacements are specified). For example, it is tempting to impose (in its strong form) RSF scaling: for a wedge, as shown by Samsioe (1955) and Cantelaube & Goddard (1997) this reduces the admissible solutions to a piecewise linear form. Such a postulate may seem quite harmless: after all, we have emphasized already that the observations do themselves show (weak) RSF scaling. However, according to these models, the central part of the pile can correctly be viewed as an elastic continuum; hence from any solution for the stresses it *should be* physically possible to generate another by an infinitesimal pushing and pulling of the elastic material along the rough base. Accordingly one has no reason to expect even the weak RSF scaling observed experimentally. Setting this aside, one could impose weak RSF scaling by assuming a basal displacement field of the same overall shape for piles of all sizes. However, as pointed out by Evesque (1997), even this imposition does not require the *strong* form of RSF scaling assumed by Cantelaube & Goddard (1997). In summary, simple elastoplastic models of sandpiles *require* that the experimental results for the force at the base depend on how the material was previously manipulated. Any attempt to predict the forces without specifying these manipulations is misguided.

A third reaction, therefore, is to start modelling explicitly the physical processes going on at the base of the pile. As mentioned previously, one is required to specify a displacement field at the base of the elastic zone; more accurately, it is the product of the displacement field and the elastic modulus that matters. This need not vanish in the large modulus limit (Section 3b); one possible choice, nonetheless, is to set the displacement field to zero at a finite modulus (which might then be taken to infinity). The simplest interpretation of this choice is by appeal to a model in which the 'sandpile' is constructed as follows (Fig. 5(a)): an elastoplastic wedge, floating freely in space, is brought to rest in contact with a rough surface, in a state of zero strain. Once in contact, gravity is switched on with no further adjustments in the contact region allowed. This might be referred to as the 'spaceship model' (or perhaps the 'floating model') of a sandpile. This illustrates two facts: (a) in considering explicitly the displacement field at the bottom surface, elastoplastic modellers are obliged to make definite assumptions about the previous history of the material; (b) these assumptions do not usually have much in common with the actual construction history of a sandpile made by pouring. A possible alternative to the spaceship model, in which unstressed laminae of elastoplastic material are successively added to an existing pile (Fig. 5(b)) is discussed in Appendix A.

(*d*) *Determinacy and anisotropy*

We shall now show that hyperbolic behaviour can be recovered from an elastoplastic description by taking a strongly anisotropic limit (Cates *et al.*, to be published). For simplicity we restrict attention to the FPA model.

The FPA model describes, by definition, a material in which the shear stress must vanish across a pair of orthogonal planes fixed in the medium – those normal to the (fixed) principal axes of the stress tensor. According to the Coulomb inequality, which the model also adopts, the shear stress must also be less than $\tan\phi$ times the normal stress, across planes oriented in all other directions. Clearly this combination of requirements can be viewed as a limiting case of an elastoplastic model with an anisotropic yield condition:

$$|\sigma_{tn}| \leq \sigma_{nn}\ \tan\Phi(\theta) \tag{4.1}$$

where θ is the angle between the plane normal $\mathbf{n}$ and the vertical (say). An anisotropic yield condition should arise, in principle, in any material having a nontrivial fabric, arising from its construction history. The limiting choice corresponding to the FPA model for a sandpile is $\Phi(\theta) = 0$ for $\theta = (\pi - 2\phi)/4$ (this corresponds to planes where $\mathbf{n}$ lies parallel to the major principal axis), and $\Phi(\theta) = \phi$ otherwise. (There is no separate need to specify the second, orthogonal plane across which shear stresses vanish, since this is assured by the symmetry of the stress tensor.) By a similar argument, all other OSL models can also be cast in terms of an anisotropic yield condition, of the form $|\sigma_{tn} - \sigma_{nn}\ \tan\Psi(\theta)| \leq \sigma_{nn}\tan\Phi(\theta)$ where $\Phi(\theta)$ vanishes, and $\Phi(\theta)$ is finite for two values of θ. (This fixes a *nonzero* ratio of shear and normal stresses across certain special planes.)

At this purely phenomenological level there is no difficulty in connecting hyperbolic models smoothly onto *highly anisotropic* elastoplastic descriptions. Specifically, consider a medium having an orientation-dependent friction angle $\Phi(\theta)$ that does not actually vanish, but is instead very small ($\leq \epsilon$, say) in a narrow range of angles (say of order ϵ) around $\theta = (\pi - 2\phi)/4$, and approaches ϕ elsewhere. (One interesting way to achieve the required yield anisotropy is to have a strong anisotropy in the *elastic* response, and then impose a *uniform* yield condition to the strains, rather than stresses.)

Such a material will have, in principle, mixed elliptic/hyperbolic equations of the usual elastoplastic type. The resulting elastic and plastic regions must nonetheless arrange themselves so as to obey the FPA model to within terms that vanish as $\epsilon \to 0$. If ϵ is small but finite, then for this elastoplastic model the results will depend on the basal boundary condition, but only through these higher order corrections to the leading (FPA) result. We show in Section 6 below that the case of small but finite ϵ is exactly what one would expect if a small amount of particle deformability were introduced to a fragile skeleton of rigid particles obeying the FPA constitutive relation.

Although somewhat contrived (from an elastoplastic standpoint), the above choice of anisotropic yield condition establishes an important point of principle, and may point toward some important new physics. Although elastoplastic models do suffer from elastic indeterminacy (they require a basal displacement field to be specified), the extent of the influence of the boundary condition on the solution depends on the model chosen. Strong enough (fabric-dependent) anisotropy, in an elastoplastic description, might so constrain the solution that, although it

suffers elastic indeterminacy in principle, it does so only harmlessly in practice. Under such conditions it is *primarily the fabric* and only minimally the boundary conditions which actually determine the stresses in the body. For models such as that given above there is a well-defined limit where the indeterminacy is entirely lifted, hyperbolic equations are recovered, and it is quite proper to talk of local stress propagation 'rules' which are determined, independently of boundary conditions, by the fabric (hence construction history) of the material.

Our modelling framework, based precisely on these assumptions, will be valid for sandpiles if, as we contend, their physics lie close to this limit of 'fabric dominance' (see Section 6 below). This contention is consistent with, though it does not require, belief in the existence of an underlying elastoplastic continuum description.

5. Experimental results

Before discussing in more detail the physical interpretation of our models, we give a brief account of the experimental data. In doing this, it is important to draw a distinction between (axially symmetric) cones, and (translationally symmetric) wedges of sand. The latter is a quasi-two dimensional geometry. The main question is, to what extent the pressure-dip can be trusted as a reproducible experimental phenomenon for a sandpile constructed by pouring onto a rough rigid support. In particular, Savage (1997) has drawn attention to the possible role of small deflections in the base ('basal sag') in causing the dip to arise.

(*a*) *Cones*

The earliest data we know of, on conical sandpiles, is that of Hummel & Finnan (1920) who observed a pronounced stress dip. However, their pressure cells were apparently subject to extreme hysteresis, and these results cannot be relied upon. Otherwise the only data prior to Smid & Novosad (1981) *for cones* is that of Jokati & Moriyami (1979). Although a stress dip is repeatedly observed by these authors, their results (on rather small piles) do not show consistent RSF scaling. The well-known data of Smid & Novosad (1981) shows a clear stress minimum at the centre of the pile. Even this dataset is not completely satisfactory: the observation of the dip is based on the data from a single (but calibrated) pressure cell beneath the apex. However, the data for different pile heights shows clear (weak) RSF scaling, and is quantitatively fit by the FPA model with either of the secondary closures shown in Fig. 1. Savage (1997) points out that 'it is not possible from the information given to estimate the deflections [at the base] that might result from the weight of the pile'. Smid and Novosad, however, describe their platform as 'rigid'. †

Recently, Brockbank *et al.* (1997) have performed a number of careful measurements on relatively small piles of sand (as well as flour, glass beads, etc.). The pressure transducers comprise an assembly of steel ball-bearings lying atop a thin blanket of transparent rubber on a rigid glass plate; material is poured from

† Savage (1997) also criticises the reduction method used to analyse this data by Wittmer *et al.* (1997*a*), as shown in Fig. 1; when normalizing stresses by the mean density of the pile, he apparently prefers to use a separate measurement of the bulk density (in a different geometry), rather than the density deduced by integrating the vertical normal stresses to give the weight of the pile.

a point source onto this assembly. The deflection of the ball-bearings is estimated as 10 μm. By calibrating and optically monitoring their imprints on the rubber film, the vertical stresses can be measured. Perhaps the most interesting feature of this method is that, although the basal deflection is certainly not zero, it is of a character quite unlike basal sag. Indeed, the supporting ball-bearings are deflected downward (indenting the rubber film) in a manner that depends on the *local* compressive stress, as opposed to the cumulative (*i.e.*, nonlocal) effect of sagging. The latter is bound to be maximal under the apex of the pile, whereas the indentation is maximal under the zone of maximum vertical compressive stress, wherever that may be. If the stress pattern is controlled by slight deformations of the base, there would be no reason to expect a similar stress pattern to arise for an indentable base, as for a sagging one.

But in fact, a very similar stress pattern is seen (Fig. 1). The data shown here involve averaging over several piles, since the setup measures stresses over quite small areas of the base (the ball bearings are 2.5 mm diameter) and these stresses fluctuate locally, as is well-known (Liu *et al.* 1995; Claudin & Bouchaud 1997). Although still subject to relatively large statistical scatter, the data show an unambiguous dip of very similar magnitude to that reported by Smid and Novosad; moreover the dip is spread over several, rather than a single, transducer(s).

It is, of course, important to distinguish conceptually the noisiness of this data (arising from fluctuations at the granular level) from any intrinsic irreproducibility of the results. If the results are reproducible, then for large enough piles one might expect the averaging over several piles to be obviated by binning the data over many transducers. This is, in effect, what Smid and Novosad do (since their transducers are much larger). More careful experimental investigations of this point would, nonetheless, be welcome.

We conclude from this recent study, which substantially confirms the earlier work of Smid & Novosad (1981), that the attribution of the stress dip to basal sag is not justified for the case of conical piles of sand. Brockbank *et al.* (1997) also saw a stress dip for small, but not large, glass beads. This difference suggests that to observe the dip requires a large enough pile compared to the grain size – perhaps to allow an anisotropic mesoscale texture to become properly established. No dip was seen for lead shot (deformable) or flour (cohesive).

(*b*) *Wedges*

The experiments on *wedges* appear very different. The papers of Hummel & Finnan (1920), and Lee & Herington (1971) include datasets for which the construction history is described as being effectively from a line source. These results, as well as others cited by Savage (1997) offer support for his conclusions (made earlier by Burman & Trollope 1980) that the construction history of the wedge does not much matter, and that there is only a very small or negligible dip for wedges supported by a fully rigid base. These studies also suggest that a dip appears almost immediately if the base under the wedge is allowed to sag. These results, if confirmed by careful repetition of the experiments, would certainly cast doubt on FPA-type models as applied to wedges.

(*c*) *Specifying the construction history*

Such historic experiment, measuring the stress distribution for wedges made supposedly from a line source, need careful repetition. This is because, even from

a point source (conical pile) or line source (wedge) at least two different types of construction history are possible. The first is when, as assumed in FPA-type models, the grains avalanche in a thin layer down the free surface. The second, which, like the first, has clearly been observed in three-dimensional work on silo filling (Munch-Andersen & Nielsen 1990, 1993) is called 'plastic cone' behaviour. It entails the impacting grains forcing their way downwards at the apex into the body of the pile, which then spreads sideways. A parcel of grains arriving at the apex ends up finally as a thin horizontal layer. (A transition between this and surface avalanche flow may be controlled by varying the height from which grains are dropped, among other factors.) A third possibility is that of 'deep yield' (see Evesque & Boufellouh 1997): a buildup of material near the apex followed by a deep avalanche in which a thick slab of material slumps outwards (Evesque 1991).

These different construction histories, even among piles created from a point or line source, would lead one to expect quite different stress patterns. For example, the plastic cone construction should lead to a texture with local symmetry about the vertical, as assumed by Bouchaud *et al.* (1995). This model, which we also expect to describe a conical pile built by sieving sand uniformly onto a supporting disc (Wittmer *et al.* 1997*b*) does not give a pressure dip. Although in point-source experiments on cones the surface avalanche mechanism is usually seen (Evesque 1991; Evesque *et al.* al. 1993) we do not know whether the same applies for wedges; the classical literature is ambiguous (Hummel & Finnan 1920; Lee & Herington 1971). For these reasons such experiments must be repeated, with proper monitoring of the construction history, before conclusions can be drawn.

There are, in fact, good reasons why the surface avalanche scenario, on which models such as FPA depend, may be very hard to observe in the wedge geometry. Recall that for the wedge geometry at repose, all OSL models predict an outer sector of the wedge, of substantial thickness, in which the Coulomb inequality is saturated. Clearly, if avalanches take place on top of a thick slab of material already at incipient failure, it may be impossible to avoid rearrangements deeper within the pile, leading either to 'deep yield' or 'plastic wedge' behaviour. To this extent the application of FPA-type models to a wedge geometry is not necessarily self-consistent. The same does not apply in the conical geometry, where the solution of these models predicts only an infinitesimal plastic layer at the surface of the cone (Wittmer *et al.* 1996). Accordingly it would be very interesting to compare experimentally wedges and cones of the same material to see whether the character of the avalanches is fundamentally different, as FPA-like models might lead one to expect. Further experiments involving comparison of histories are suggested by Wittmer *et al.* (1997*a*, *b*).

Although there are, so far, few data showing a clear dependence of measured stresses on construction history in freestanding cones or wedges, the effect is well-established in experiments on silos. Specifically, for flat-bottomed silos filled by surface avalanches from a point source, the vertical normal force at the centre of the base is less than at the edge (Munch-Andersen & Nielsen 1990). This effect, which is readily explained within an FPA-type modelling approach (Wittmer *et al.* 1997*a*), is not reported in silos filled by sieving, nor when a plastic cone behaviour is seen at the apex (Munch-Andersen & Nielsen 1993).

6. Sandpiles as fragile matter

As we have emphasized, the continuum mechanics represented by our hyperbolic models is not that of conventional elastoplasticity. In what follows we develop an outline interpretation of this continuum mechanics as that appropriate to a material in which stresses propagate primarily along force chains. Simulations of frictional spheres offer some support for the force-chain picture, at least as a reasonable approximation: most of the deviatoric stress is found to arise from *strong, normal* forces between particles participating in force chains; tangential forces (friction) and the weaker contacts transverse to the chains contribute mainly to the isotropic pressure (Thornton & Sun 1994; Thornton 1997; C. Thornton, this volume). In addition to this, the content of our models is to assume that the skeleton of force chains is *fragile*, in a specific sense defined below.

(*a*) *Force chains*

Informally speaking, the hyperbolic problem posed by OSL models is determined once half of the boundary forces are specified. More precisely (Fig. 3(b)) one is required to specify the surface force tangential to each characteristic ray, at one end and *one end only.* The corresponding force acting at the other end is obliged to balance the sum of the specified force, any body forces acting tangentially along the ray. If it does not do so, then within our modelling approach, the material ceases to be in static equilibrium. This is no different from the corresponding statement for a fluid or liquid crystal; if boundary conditions are applied that violate the conditions for static equilibrium, some sort of motion results. Unlike a fluid, however, for a granular medium we expect such motion to be in the form of a finite rearrangement rather than a steady flow. Such a rearrangement will change the microtexture of the material, and thereby *alter the constitutive relation among stresses.* We expect it to do so in such a way that the new network of force chains (new constitutive relation) is able to support the newly imposed forces.

Although simplified, we believe that this picture correctly captures some of the essential physics of force chains. Such chains are load-bearing structures within the contact network and, in the simplest approximation of straight chains of uniform orientation these must have the property described above: any difference in the forces on two ends of a path must be balanced by a body force. Note that if one makes a linear chain of more than two rigid particles with point contacts, then to avoid torques, this can indeed support only tangential forces, regardless of the local friction coefficient between the grains themselves; see figure 6(a). Force chains should, we believe, be identified (on the average) with the characteristic rays of our hyperbolic equations. The *mean orientation* of the force chains is then reflected in a constitutive equation such as FPA or OSL.

Our modelling approach thus assumes that the mean orientation of force chains, in each element the material, is fixed at burial. (This does not necessarily require that the individual chains are themselves fixed.) We think it reasonable to assume that the force chains will not change their average orientations so long as they are able to support subsequent applied loads. But if a load is applied which they cannot support (one in which the tangential force difference and body force along a path do not match) irreversible rearrangement is inevitable (Evesque, private communication). This causes some part of the pile to adopt a new microtexture

and thereby a new constitutive relation. In other words, *incompatible* loadings of this kind must be seen as part of the construction history of the pile.

(*b*) *Fragile matter*

There is a close connection between these ideas and recent work on the 'marginal mechanics' of periodic arrays of identical grains. (This is considered further in Appendix B.) The marginal situation is where the (mean) coordination number of the grains is the minimum required for mechanical integrity; in two dimensions this is three for frictional and four for frictionless spheres. (Larger coordination numbers are needed for aspherical grains.) Indeed, each OSL models rigorously describes the continuum mechanics of a certain ordered array of this kind (see Appendix B). Marginal packings are exceptional in an obvious sense: most packings of grains one can think of do not have this property, and the forces acting on each grain cannot be found without further information. However, we can interpret this correspondence between continuum and discrete equations, not at the level of the packing of individual grains (for which the marginal coordination state would be hard to explain) but at the level of a granular skeleton made of force chains. The OSL models (in two dimensions) can then be viewed as postulating a simplified, marginally stable geometry of the skeleton, in which a regular lattice of force chains (bearing tangential forces only) meet at four-fold coordinated junctions. (For the FPA model, though not in general, this lattice is rectangular. See figure 6(b).) Such a skeleton leads to hyperbolic equations (or perhaps parabolic ones if enough disorder is added); its mechanics are determinate in the absence of a displacement field specified at the base.

In the present context, fragility arises from the the requirement of tangential force balance along force chains. If this is violated at the boundary (within the models as so far defined, even infinitesimally) then internal rearrangement must occur, causing new force chains to form, so as to support the load. It seems reasonable to assume that when rearrangements are forced upon the system, it responds in an 'overdamped manner' – that is, the motion ceases as soon as the load is once again supported. If so, one expects the new state to again be marginally stable. This suggests a scenario in which the skeleton evolves dynamically from one fragile state to another. By such a mechanism, marginally stable packings, although exceptional in the obvious sense that most packings one can think of are not marginal, may nonetheless be generic in unconsolidated dry granular matter. Thornton (1997) reports that, in simulations of frictional spheres, force chains do rearrange strongly under slight reorientations of the applied load.

Consider finally a regular lattice of force chains, for simplicity rectangular (the FPA case) which is fragile if the chains can support only tangential loads. This is the case so long as such paths consist of linear chains of rigid particles, meeting at frictional point contacts: as mentioned above, the forces on all particles within each chain must then be colinear, to avoid torques. This imposes the (FPA) requirement that there are no shear forces across a pair of orthogonal planes normal to the force chains themselves (see Section 3*d*). Suppose now a small degree of particle deformability is allowed (Cates *et al.*, to be published). This relaxes *slightly* the collinearity requirement, but only because the point contacts are now flattened. The ratio ϵ of the maximum transverse load to the normal one will therefore vanish as some power of the mean deformation. This yield criterion applies only across two special planes; failure across others is governed

by some smooth yield requirement (such as the ordinary Coulomb condition: the ratio of the principal stresses lies between given limits). The granular skeleton just described, which was fragile in the limit of rigid grains, is now governed by a strongly anisotropic elastoplastic yield criterion of precisely the kind described in Section 3d. The skeleton can support loads that do violate the tangential balance condition, but only through terms that vanish as $\epsilon \to 0$. To escape the hyperbolic regime of 'fabric dominance', ϵ must be significant, which in turn requires significant particle deformation under the influence of the mean stresses applied.

This indicates how a non-fragile packing of frictional, deformable rough particles, displaying broadly conventional elastoplastic features when the deformability is significant, can approach a fragile limit when the limit of a large modulus is taken at fixed loading. (It does not, of course, imply that *all* packings become fragile in this limit.) Conversely it shows how a packing that is basically fragile (in its response to gravity) could nonetheless support very small incremental deformations, such as sound waves, by an elastic mechanism. The question of whether sandpiles are better described as fragile, or as ordinarily elastoplastic, remains open experimentally. To some extent it may depend on the question being asked. However, we have argued, on various grounds, that in calculating the stresses in a pile under gravity a fragile description may lie closer to the true physics.

7. Conclusion

From the perspective of geotechnical engineering, the problem of calculating stresses in the humble sandpile may appear to be of only of marginal importance. The physicist's view is different: the sandpile is important, because it is one of the simplest problems in granular mechanics imaginable. It therefore provides a test-bed for existing models and, if these show shortcomings, may suggest ideas for improved physical theories of granular media.

There are, in physics, certain types of problem for which the fundamental principles or equations are clear, and the difficulty lies in working out their consequences. An example is the use of the Navier Stokes equation in studies of (say) turbulence. The form of the Navier Stokes equation can be deduced by considering only the symmetries and conservation laws of an isotropic fluid. Accordingly, its status is not, as sometimes assumed, that of an approximation based on constitutive hypotheses that happen to be very accurate for certain materials. Rather, it describes a limiting behaviour, which all members of a large class of materials (viscoelastic fluids included) approach with indefinite accuracy in the limit of long length- and time-scales. (We are aware of no theory of elastoplasticity having remotely similar status.) There are other types of problem in which the fundamentals are not clear. For such problems, the governing equations must first be established, before they can be solved. We remain convinced that the static modelling of *poured assemblies of cohesionless grains under gravity* is of this second type. This view is not particularly new, either among physicists (Edwards & Oakeshott 1989), or among engineers (Gudeshus 1985,1997).

From this perspective, we can see no reason why the starting points of simple rigid-plastic or elastoplastic continuum mechanics should offer significant insights into the sandpile problem. Simple elastoplastic approaches, in particular, give

only one unambiguous physical prediction: that a sandpile supported by a rough base should have *no definite behaviour*. Experimentalists, who believe themselves to be measuring a definite result, are likely to be baffled by such predictions. For if, as these models require, the forces acting at the base of a pile can be varied at will without causing its static equilibrium to be lost (by making small elastic displacements at the base), then all the published 'measurements' of such forces must be dismissed as artefact. An alternative view is that these represent rather haphazard investigations of some unspecified physical mechanism that does somehow determine a displacement field at the base of the pile. (As mentioned previously, basal sag is certainly not an adequate candidate.) The challenge of whether, for cohesionless poured sand, such a displacement field can sensibly be defined, remains open.

Given the present state of the data, a conventional elastoplastic interpretation of the experimental results for sandpiles may remain tenable; more experiments are urgently required. In the mean time, a desire to keep using tried-and-tested modelling strategies until these are demonstrably proven ineffective is quite understandable. We find it harder to accept the suggestion (Savage 1997) that anyone who questions the complete generality of traditional elastoplastic thinking is somehow uneducated.

Our own position is not that elastoplasticity itself is dead, but we do believe that macroscopic stress propagation in sandpiles is determined much more by the internal fabric of the material (therefore the construction history) and much less by boundary conditions, than *simple* elastoplastic models suggest. Reasons for this, based on the idea of a fragile skeleton of force chains, have been discussed above. By considering a particular form of yield condition, we have shown how a fragile model can be matched smoothly onto a relatively conventional, but strongly anisotropic, elastoplastic theory. Thus it is possible in principle to have a model which, although strictly governed by the mixed hyperbolic/elliptic equations of elastoplasticity, leads to solutions that obey purely hyperbolic equations everywhere, to within (elastically indeterminate) corrections that are small in a certain limit. In such a system the results will depend less and less on boundary conditions, and more and more on fabric, as that limit is approached. Moreover, for certain well-defined fragile packings of frictional grains, the limit is the rigid particle one, in which the elastic modulus of the grains is taken to infinity at fixed loading.

In summary, we have discussed a new class of models for stress propagation in granular matter. These models assume local propagation rules for stresses which depend on the construction history of the material and which lead to hyperbolic differential equations for the stresses. As such, their physical basis is substantially different from that of conventional elastoplastic theory (although they may have much more in common with 'hypoplastic' models). Our approach describes a regime of 'fragile' behaviour, in which stresses are supported by a granular skeleton of force chains that must undergo finite internal rearrangement under certain types of infinitesimal load. Obviously, such models of granular matter might be incomplete in various ways. Specifically we have discussed a possible crossover to elastic behaviour at very small incremental loads, and to conventional elastoplasticity at very high mean stresses (when significant particle deformations arise). However, we believe that our approach, by capturing at least some of the physics of force chains, may offer important insights that lie beyond the scope of conven-

tional elastoplastic or rigid-plastic modelling strategies. The equivalence between our fragile models and limiting forms of extremely *anisotropic* elastoplasticity, has been pointed out.

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Appendix A. Laminated elastoplastic cone

As an alternative to the 'spaceship model', one might envisage (Fig. 5(b)) the creation of a pile by incremental addition of thin layers of elastoplastic material to its upper surface (in imitation of an avalanche). It might then be argued that this thin layer, being under negligible stress, must be characterized by a zero displacement field (Savage 1998). On a rough support, one would then expect the displacement at the base to remain zero as further additions to the pile are made, giving a zero displacement boundary condition at the base of what has, by now, presumably become a simple elastoplastic body.

This reasoning is flawed: the same argument entails that, at any stage of the pile's construction, the *last* layer added is in a state of zero displacement, not just where it meets the base, but along its entire length. If so, then not only the base but also the free surface of the pile is subject to a zero displacement boundary condition. For a simple elastoplastic cone or wedge, this is incompatible with the zero stress boundary condition already acting at the free surface. (Such a body, in effect suspended under gravity from a fixed upper surface, will exert forces across that surface, as well as across the supporting base).

The paradox is resolved by noticing that this 'laminated elastoplastic' model in fact involves the addition of thin, stress-free elastoplastic layers to an already deformed body. The result will not be a simple elastoplastic continuum, but a body in which internal stresses and displacements are present even when all body forces are removed (like a reinforced concrete pillar, or a tennis racket made of laminated wood) – Fig.5(b). Such a body can, if carefully designed with a specific loading in mind, satisfy simultaneously a zero stress and zero displacement (more properly, constant displacement) boundary condition at any particular surface. These rather intriguing properties may well be worth investigating further, but they are still a long way from a realistic description of the construction history of a sandpile. In any case it is misleading to suggest (Savage 1998) that such considerations can justify the adoption of a zero displacement basal boundary condition within an ordinary (*i.e.*, not pre-strained), isotropic elastoplastic continuum model.

Appendix B. Microscopic force transmission models

Note first that a very large class of discrete models lead directly to OSL models in the continuum limit. A simple example is defined in Fig. 7(a). As shown by Bouchaud *et al.* (1995), this model gives a wave equation with two characteristic rays symmetrically arranged about the vertical axis. If the symmetry in the stress propagation rules is broken, an asymmetric OSL model arises instead (Fig. 7(b)).

Secondly, when the continuum limit of such force-transfer models is taken, one has (in two dimensions) *only two characteristic rays* even if the force transfer rules

involve more than two neighbours in the layer below. An example (Claudin *et al.* 1998) is shown in Fig. 7(c). Broadly speaking, one recovers an OSL model, in the continuum limit, whenever the forces passed from a grain to its downward (or sideways) neighbours obey a deterministic linear decomposition of the 'incident force' (f_x, f_z), defined as the vector sum of the forces acting from grains in the layer above, plus the body force on the given grain.

Trollope's model, whose force transfer rules are as shown in Fig. 7(d-f), is not a member of this class. (Indeed it has three characteristic rays in the continuum limit, rather than two.) This is because *the vector sum of the incident forces on a grain is not taken* before applying a rule to determine the outgoing forces from that grain; the latter depend *separately* on each of the incident forces. As a description of hard frictional grains, we consider this unphysical. For, if the grain in Fig. 7(d) is subjected to two equal small extra forces f from its two neighbours in the layer above (whose vector sum is vertical) the net effect on the outgoing forces should be equivalent to a small increase in its weight $w = 2f\cos\theta$. Within Trollope's model, this is not the case. Since its propagation rules are linear, any attempt to rectify this feature (by taking the vector sum of the forces before propagating these on to the next layer) will give an OSL model instead.

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Figure 1. Comparison of FPA model using a uniaxial secondary closure (Wittmer *et al.* 1996, 1997*a*) with scaled experimental data of Smid & Novosad (1981) and (*) that of Brockbank *et al.* (1997) which was averaged over three piles. Upper and lower curves denote normal and shear stresses. The data is used to calculate the total weight of the pile which is then used as a scale factor for stresses. The horizontal coordinate is scaled by the pile radius.

Figure 2. Vertical normal stress found for the BCC model described in Section 2d , for a pile at angle of repose $\phi = 30$ degrees, compared to the active and two passive IFE solutions (obtained by shooting from the midplane) discussed in the text. (Out of numerical reasons the continuous IFE uses $P = (\sigma_{zz} + \sigma_{xx})/2$ and the polar angle θ as functions of the direction of the principal axis Ψ.) Note that active and passive IFE solutions do not bound the stress, either in the BCC model or in the simple elastoplastic model of Cantelaube & Goddard (1997), which, for a certain parameter choice, yields identical results. The 2-dimensional FPA solution is also included (dotted line).

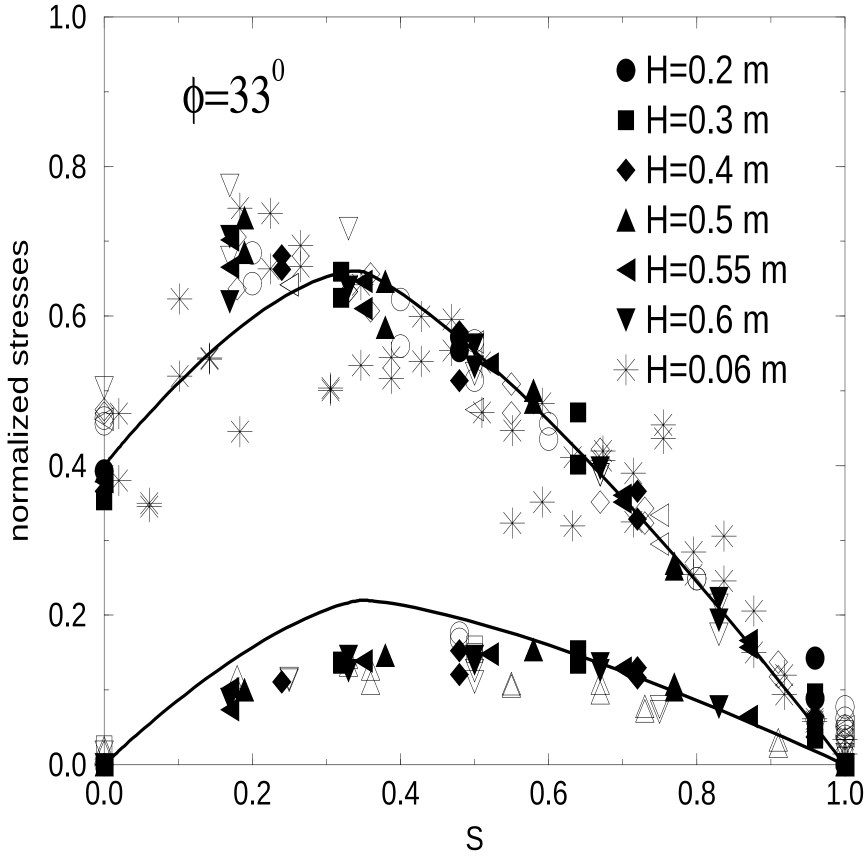
Figure 3. (a) The response to a localized force is found by resolving it along characteristics through the point of application, propagating along those which do not cut a surface on which the relevant force component is specified. For a pile under gravity, propagation is only along the downward rays. (b) Admissible boundary conditions cannot specify separately the force component at both ends of the same characteristic. If these forces are unbalanced (after allowing for body forces), static equilibrium is lost.

Figure 4. Starting from an elastic cone or wedge on a rough support, any initial stress distribution can be converted to another by displacements with respect to the rough ‘pinning’ surface (a) $\rightarrow$ (b). Taking the limit of a high modulus (b) $\rightarrow$ (c) at fixed surface forces, an arbitrary stress field remains, while recovering the initial shape of the cone and satisfying the free surface boundary conditions. This shows the physical character of ‘elastic indeterminacy’ for an elastic or elastoplastic body on a rough support.

Figure 5. (a) The ‘spaceship’ model of a sandpile. An unstrained, isotropic elastoplastic cone or wedge is brought into contact with a rough surface and gravity then switched on. (b) ‘Laminated elastoplastic model’ of a sandpile. Layers are added in a state of zero stress (thereby, it is argued, zero displacement) to a pre-existing, gravitationally loaded pile. Such a pile (if gravity is removed) will spring into a new shape, characterized by a nonzero internal stress field (Appendix B).

Figure 6. (a) A force chain (“stress path”) of hard particles can support only tangential compressive loads in static equilibrium. This is to avoid torques on particles in the chain (gravitational torques acting directly on the particles within a chain are ignored). (b) A simple realization of the FPA model as a rectilinear arrangement of force chains under tangential loading.

Figure 7. (a) Force transfer rules for a simple discrete model (Hong 1993, Bouchaud *et al.* 1995). The forces obey $(f_2 - f_1)\sin\theta = f_x$ and $(f_1 + f_2)\cos\theta = f_z + w$ with w that of gravity. A first order difference equation for f_x is found by writing $f_x(x,z) = [f_2(x-\Delta x, z-\Delta z) - f_1(x+\Delta x, z-\Delta z)]\sin\theta$, with $\Delta x = d\cos\theta$ and $\Delta z = d\sin\theta$, and eliminating $f_{1,2}$ in favour of $f_{x,z}$ (d is the grain diameter). A similar procedure is then followed for f_z. In the continuum limit, the resulting first order differential equations give the BCC model (with $c_0 = \tan\theta$) with two characteristics (right). (b) The same, with asymmetric propagation rules, leading to an asymmetric OSL model. (c) A simple model with three downward neighbours. The force assignment rule for the middle ray is $f_2 = \alpha(f_1 + f_3)$, where α is some constant. As shown by Claudin *et al.* (1998), the result is still an OSL model (in fact BCC with $c_0 = \tan\theta' < \tan\theta$). (d) In Trollope's model, the outgoing granular forces (p', q', r') depend separately on the incoming ones (p, q, r) rather than on their vector sum: $p' - p = w/[c(1+k)]$, $q' - q = wk/[c(1+k)]$ and $r' = r + (1-k)wt/(1+k)$. Here w is the weight of a grain and c, t denote $\cos\theta, \tan\theta$. (e) As a result, for $0 < k < 1$ a symmetrical extra loading from two neighbours above whose resultant $2fc$ is directly downwards, is not equivalent to an increase (f) in grain weight $w = 2fc$.



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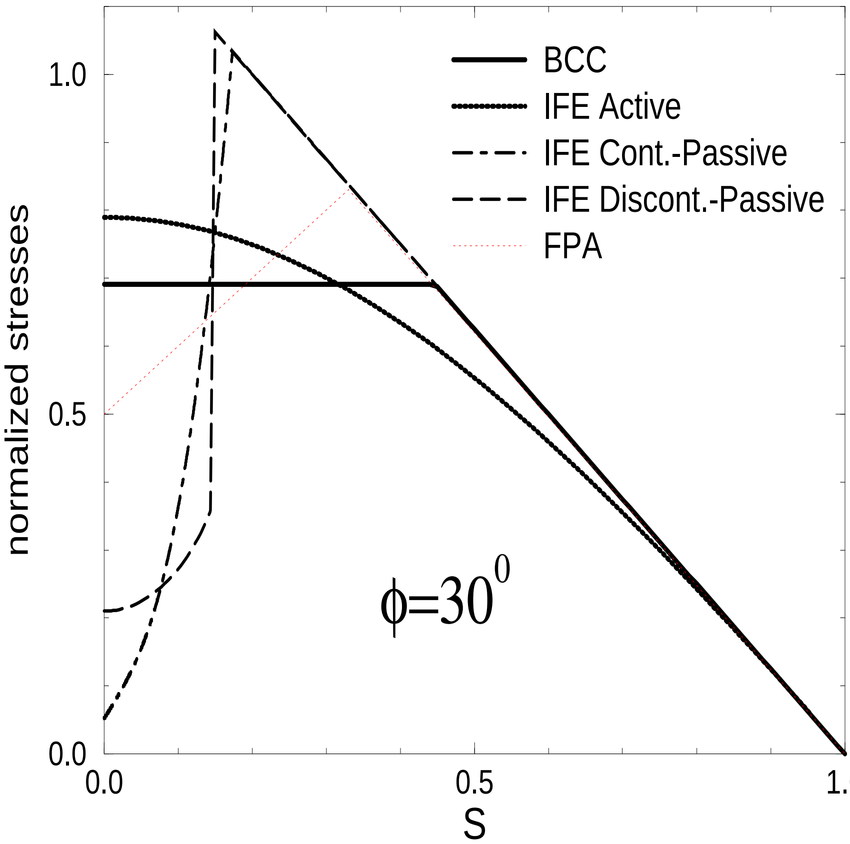

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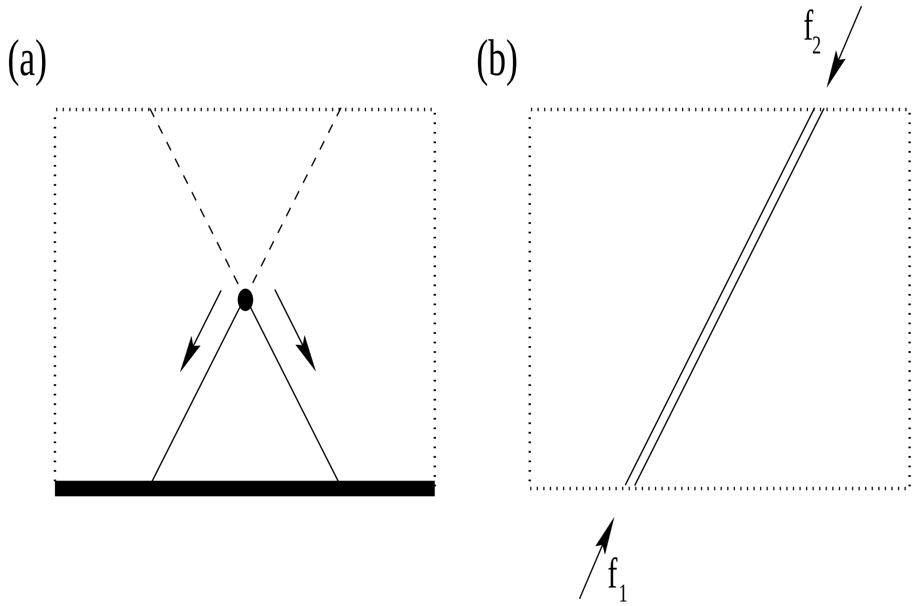

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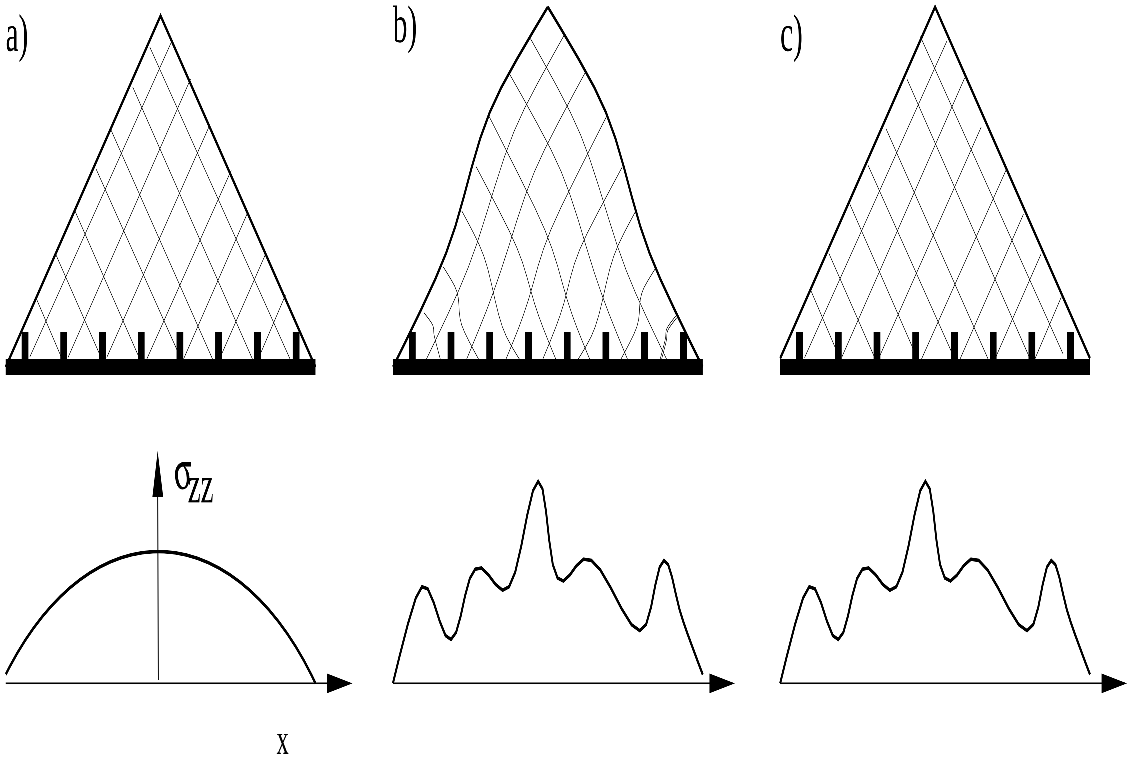

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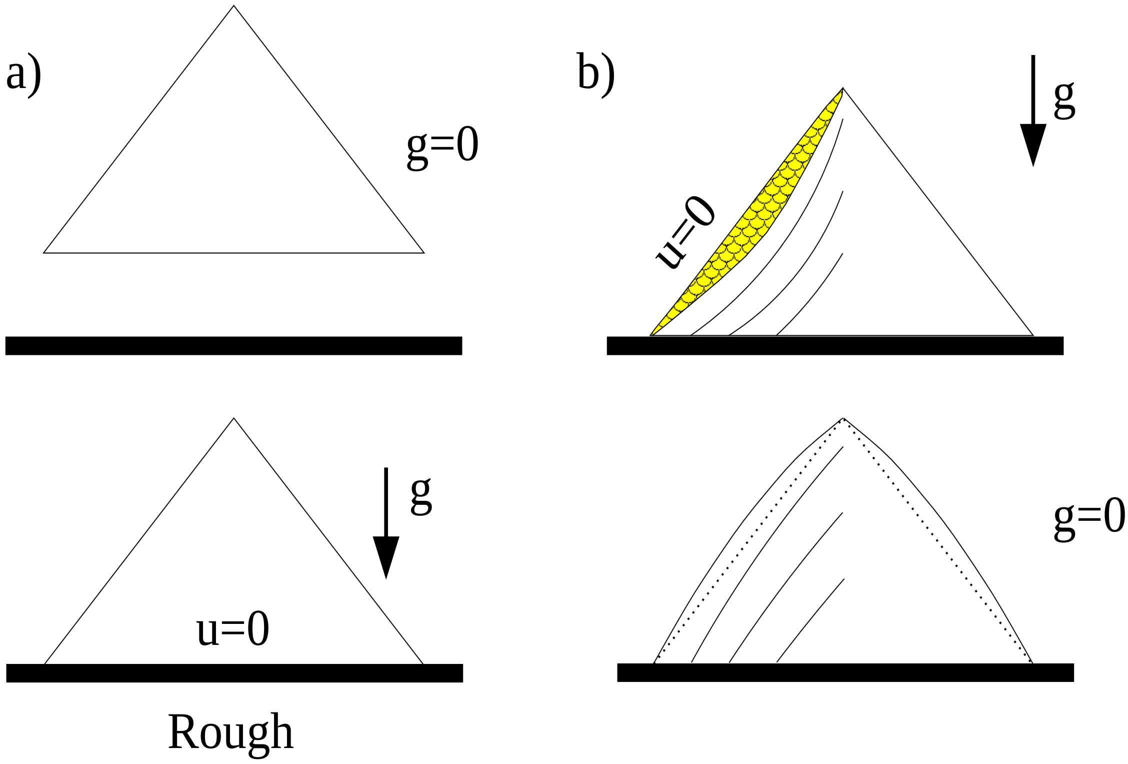

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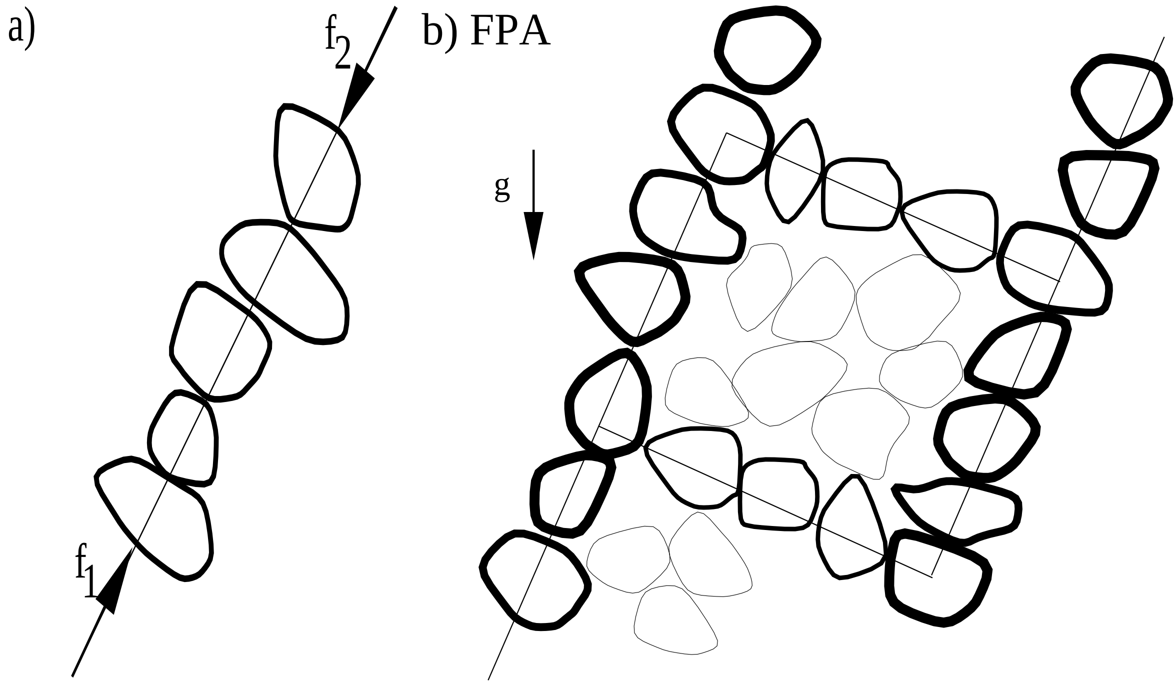

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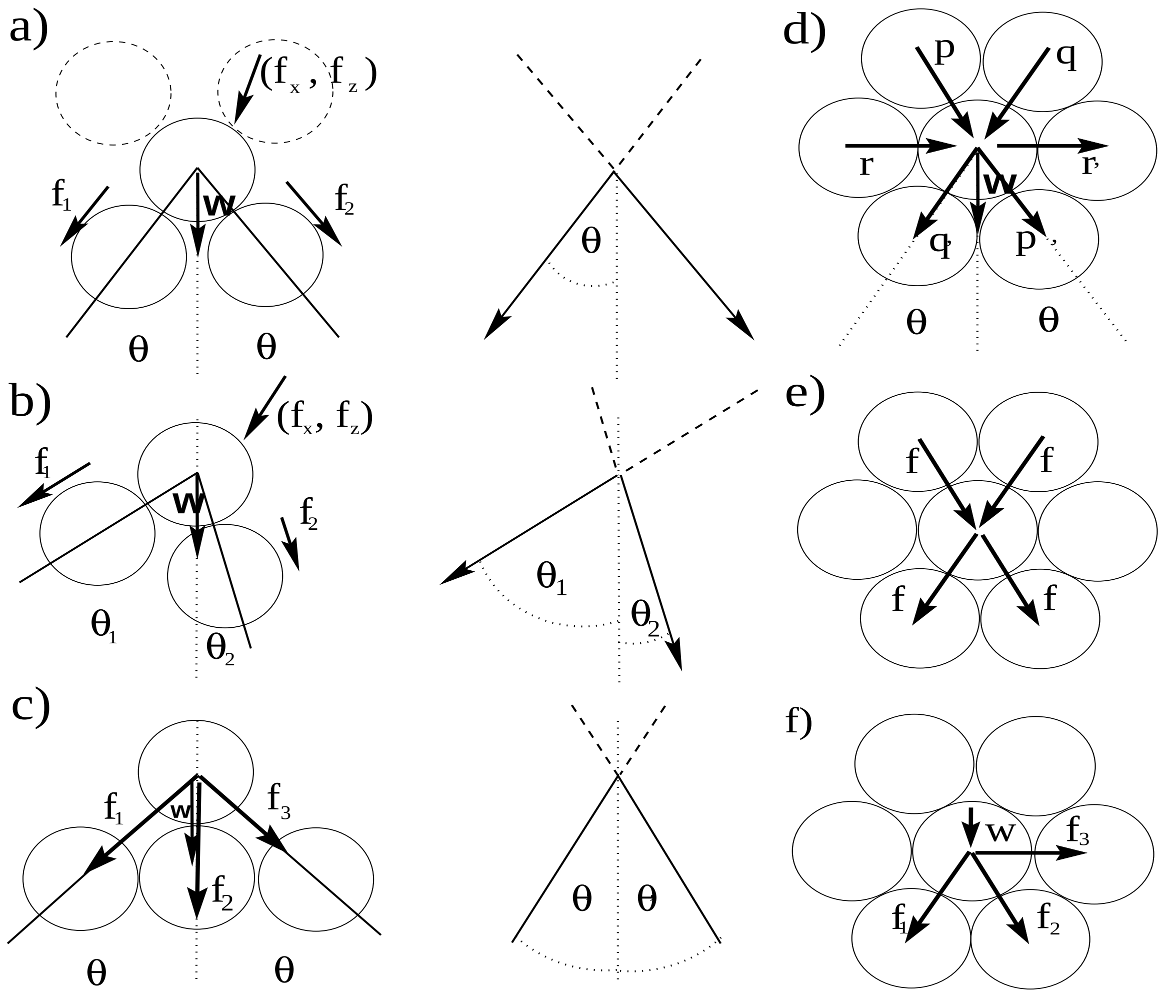

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